

# Hilbert quasi-polynomial for order domain codes

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## Abstract

We present an application of Hilbert quasi-polynomials to order domain codes, allowing the effective computation of the order domain condition in a direct way. We also provide an improved and specialized algorithm for the computation of the related Hilbert quasi-polynomials.

**Keywords:** Groebner basis, Hilbert polynomial, Hilbert quasi-polynomial, affine-variety code, order domain code.

## 1 Introduction

Fundamental algebraic invariants of a standard-graded ring can be deduced from its Hilbert-Poincaré series and Hilbert polynomial. The Hilbert quasi-polynomial generalizes Hilbert polynomial from the standard grading case to more generalized weighted grading, but until [CM16] no effective algorithms for its computation were known.

Apart from its natural ideal-theoretical application, we believe that much more practical applications can arise from its use, in particular in coding theory.

An important research area in coding theory is that of geometric codes, which are known to achieve near-optimal performance since the seminal Goppa paper ([Gop77]). A class of geometric codes that have received a lot of recent attention is formed by the so-called order-domain codes ([GP02],[Gei08]). These codes are defined by evaluating a polynomial vector space at the rational points of a related variety, and as such form a subclass of the so-called affine-variety codes ([FL98],[MOS12]). The order-domain condition depends largely on the weights of monomials under the Hilbert staircase of the positive-dimension ideal associated to the order domain. Unfortunately, this condition is very difficult to verify for arbitrary ideals and the known cases require ad-hoc proofs.

In this paper we present our application of Hilbert quasi-polynomials to order domain codes, which consists in determining the order domain condition in a direct way, once the quasi-polynomial has been computed for a related quotient ring. This computation is effective thanks to our algorithms, that improve on those of [CM16] and can be specialized to the order domain case, achieving a significant performance gain.

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The remainder of this paper is organized as follows:

- Section 2: here we provide some notation, preliminaries and known results on Hilbert functions, Hilbert quasi-polynomials, affine-variety codes and order-domain codes;
- Section 3: in this section we present some results on Hilbert quasi-polynomials, which lead to improvements in their effective calculation, and our main results Corollary 3.1 and Algorithm 3.1, which allow to decide effectively if an affine-variety code is actually an order domain code.
- Section 4: the most important family of affine-variety codes is that of codes coming from maximal curves, since these codes have large length. In this section we specialize our previous results to this family, showing some actual practical cases that can be solved easily.
- Section 5: in this section we draw our conclusions and point at possible future improvements.

## 2 Preliminaries

In this section we fix some notation and recall some known results. We denote by  $\mathbb{N}_+$  the set of positive integers, by  $\mathbb{K}$  a field, by  $R := \mathbb{K}[x_1, \dots, x_n]$  the polynomial ring in  $n$  variables over  $\mathbb{K}$ , and by  $\mathcal{M} = \mathcal{M}(X)$  the set of all monomials in the variables  $x_1, \dots, x_n$ . We assign a weight  $w_i \in \mathbb{N}_+^r$  to each variables  $x_i$ , i.e. if  $X^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathcal{M}$

$$w(X^\alpha) := w_1 \alpha_1 + \cdots + w_n \alpha_n$$

Let  $<_{\mathbb{N}_+^r}$  and  $<_{\mathcal{M}}$  be monomial orderings respectively on  $\mathbb{N}_+^r$  and  $\mathcal{M}$ , and let  $W := [w_1, \dots, w_n]$  be the vector of the variable weights. The generalized weighted degree ordering  $<_W$  defined from  $W$ ,  $<_{\mathbb{N}_+^r}$  and  $<_{\mathcal{M}}$  is the ordering given by  $X^\alpha <_W X^\beta$ , with  $X^\alpha, X^\beta \in \mathcal{M}$ , if either

$$w(X^\alpha) <_{\mathbb{N}_+^r} w(X^\beta) \text{ or } \\ w(X^\alpha) = w(X^\beta) \text{ and } X^\alpha <_{\mathcal{M}} X^\beta$$

Given  $f \in R$  and  $<_W$ ,  $\text{lm}(f)$  and  $\text{lt}(f)$  stand respectively for the leading monomial and the leading term of  $f$  w.r.t  $<_W$ . Let  $I \subseteq R$  be an ideal of  $R$ , then we denote by  $\bar{I} := \text{in}_{<_W}(I)$  the initial ideal of  $I$ , which is  $\{\text{lt}(f) \mid f \in I\}$ , and by  $\mathcal{N}_{<_W}(I)$  the Hilbert staircase of  $I$ , which is the set of all monomials that are not leading monomials of any polynomial in  $I$ . In the remainder, we suppose  $w_i \in \mathbb{N}_+$ . When  $W = [1, \dots, 1]$ , the grading is called **standard**. The pair  $(R, <_W)$  stands for the polynomial ring with the generalized weighted degree ordering  $<_W$ .

### 2.1 Introduction to Hilbert quasi-polynomials

In the following we refer to [KR05] for standard notation.

Let  $I$  be a  $W$ -homogeneous ideal. The component of  $R/I$  of degree  $k \in \mathbb{N}$  is given by

$$(R/I)_k := \{f \in R/I \mid \deg(m) = k \ \forall m \in \text{Supp}(f)\}$$

The **Hilbert function**  $H_{R/I}^W : \mathbb{N} \rightarrow \mathbb{N}$  of  $(R/I, <_W)$  is defined by

$$H_{R/I}^W(k) := \dim_{\mathbb{K}}((R/I)_k)$$

and the Hilbert-Poincaré series of  $(R/I, <_W)$  is given by

$$\text{HP}_{R/I}^W(t) := \sum_{k \in \mathbb{N}} H_{R/I}^W(k) t^k \in \mathbb{N}[[t]]$$

When the grading given by  $W$  is clear from the context, we denote respectively the Hilbert function and the Hilbert-Poincaré series of  $(R/I, <_W)$  by  $H_{R/I}$  and  $\text{HP}_{R/I}$ .

By Hilbert-Serre theorem, the Hilbert-Poincaré series of  $(R/I, <_W)$  is a rational function, that is

$$\text{HP}_{R/I}(t) = \frac{h(t)}{\prod_{i=1}^n (1 - t^{w_i})} \in \mathbb{Z}[[t]]$$

We recall that a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is a *quasi-polynomial of period  $s$*  if there exists a set of  $s$  polynomials  $\{p_0, \dots, p_{s-1}\}$  in  $\mathbb{Q}[x]$  such that  $f(n) = p_i(n)$  when  $n \equiv i \pmod{s}$ . Let  $d := \text{lcm}(w_1, \dots, w_n)$  and let  $(R/I, <_W)$  be as above.

We now refer to [Vas04] for Hilbert quasi-polynomial theory. There exists a unique quasi-polynomial  $P_{R/I}^W := \{P_0, \dots, P_{d-1}\}$  of period  $d$  such that  $H_{R/I}(k) = P_{R/I}^W(k)$  for all  $k$  sufficiently large (that we denote with  $k \gg 0$ ), i.e.

$$H_{R/I}(k) = P_i(k) \quad \forall i \equiv k \pmod{d} \quad \text{and} \quad \forall k \gg 0$$

$P_{R/I}^W$  is called the **Hilbert quasi-polynomial associated to  $(R/I, <_W)$** . Observe that if  $d = 1$  then  $P_{R/I}^W$  is a polynomial, and it is simply the **Hilbert polynomial**. We underline that the Hilbert quasi-polynomial does not depend on the chosen monomial ordering  $<_{\mathcal{M}}$ , but only on the variable weights. It consists of  $d$  polynomials, which are not necessarily distinct. The minimum integer  $k_0 \in \mathbb{N}$  such that  $H_{R/I}(k) = P_{R/I}^W(k) \forall k \geq k_0$  is called **generalized regularity index** and we denote it by  $\text{ri}^W(R/I)$ . All the polynomials of the Hilbert quasi-polynomial  $P_{R/I}^W$  have rational coefficients and the same degree  $r \leq n - 1$ , where the equality holds if and only if  $I = (0)$ . In this latter case, the leading coefficient  $a_{n-1}$  is the same for all  $P_i$ , with  $i = 0, \dots, d - 1$ , and  $a_{n-1} = \frac{1}{(n-1)! \prod_{i=1}^n w_i}$ .

## 2.2 Affine-variety codes and order domain codes

In this section, we present a class of codes, called affine-variety codes and defined in [FL98], obtained by evaluating functions in the coordinate ring of an affine variety on all the  $\mathbb{F}_q$ -rational points of the variety. Let  $I$  be any ideal in  $R := \mathbb{K}[x_1, \dots, x_n]$ , where  $\mathbb{K} := \mathbb{F}_q$  is the field with  $q$  elements. Put

$$I_q = I + (x_1^q - x_1, x_2^q - x_2, \dots, x_n^q - x_n)$$

The points of the affine variety  $\mathcal{V}(I_q)$  defined by  $I_q$  are the  $\mathbb{F}_q$ -rational points of the affine variety defined by  $I$ . Let  $\mathcal{V}(I_q) = \{P_1, P_2, \dots, P_N\}$ . Since  $I_q$  contains the polynomials  $x_i^q - x_i$  for all  $i = 1, \dots, n$ , it is a 0-dimensional and radical ideal. It follows that the coordinate ring

$$R_q := R/I_q$$

of  $\mathcal{V}(I_q)$  is an Artinian ring of length  $N$  and that there is an isomorphism  $\text{ev}$  of  $\mathbb{F}_q$ -vector spaces

$$\text{ev} : R_q \rightarrow \mathbb{F}_q^N \quad \bar{f} \mapsto (f(P_1), \dots, f(P_N))$$

where  $f$  is a representative in  $R$  of the residue class  $\tilde{f}$ . Let  $L \subseteq R_q$  be an  $F_q$ -vector subspace of  $R_q$  of dimension  $k$ . The image  $\text{ev}(L)$  of  $L$  under the evaluation map  $\text{ev}$  is called the **affine-variety code** and we denote it by  $C(I, L)$ . The dual code  $C^\perp(I, L)$  is the orthogonal complement of  $C(I, L)$  with respect to the usual inner product on  $\mathbb{F}_q^N$ .

Let  $B := \{b_1, \dots, b_k\}$  be a linear basis for  $L$  over  $\mathbb{F}_q$ , the matrix

$$\begin{pmatrix} b_1(P_1) & b_1(P_2) & \dots & b_1(P_N) \\ b_2(P_1) & b_2(P_2) & \dots & b_2(P_N) \\ \vdots & \vdots & \ddots & \vdots \\ b_r(P_1) & b_r(P_2) & \dots & b_r(P_N) \end{pmatrix}$$

is a generator matrix for  $C(I, L)$  and a parity-check matrix for  $C^\perp(I, L)$ .

**Theorem 2.1.** [FL98] *Every linear code may be represented as an affine-variety code.*

Let  $\mathcal{N}_{<_W}(I)$  be the Hilbert staircase of  $I$ . If  $\text{Supp}(b_i) \subseteq \mathcal{N}_{<_W}(I_q)$ , for all  $i = 1, \dots, k$ , and  $\text{lm}(b_1) <_W \text{lm}(b_2) <_W \dots <_W \text{lm}(b_k)$ , then the basis  $B$  is called *well-behaving* and we define  $\mathcal{L}(L) = \{\text{lm}(b_1), \dots, \text{lm}(b_k)\}$ .

Let  $\mathcal{G}$  be a Groebner basis for  $I_q$ . An ordered pair of monomials  $(m_1, m_2)$ , with  $m_1, m_2 \in \mathcal{N}_{<_W}(I_q)$ , is said to be *one-way-well-behaving (OWB)* if for all  $f$  such that  $\text{Supp}(f) \subseteq \mathcal{N}_{<_W}(I_q)$  and  $\text{lm}(f) = m_1$ , we have

$$\text{lm}(f m_2 \text{ rem } \mathcal{G}) = \text{lm}(m_1 m_2 \text{ rem } \mathcal{G})$$

where the notation " $f \text{ rem } \mathcal{G}$ " stands for the remainder of  $f$  modulo  $\mathcal{G}$ .

**Theorem 2.2.** *For any monomial ordering  $<$ , the minimum distance of  $C(I, L)$  is at least*

$$\min_{p \in \mathcal{L}(L)} |\{s \in \mathcal{N}_{<}(I_q) \mid (p, h) \text{ is OWB, } \text{lm}(ph \text{ rem } \mathcal{G}) = s\}|$$

*and the minimum distance of  $C(I, L)^\perp$  is at least*

$$\min_{s \in \mathcal{N}_{<}(I_q) \setminus \mathcal{L}(L)} |\{p \in \mathcal{N}_{<}(I_q) \mid \exists h \in \mathcal{N}_{<}(I_q) \text{ s.t. } (p, h) \text{ is OWB, } \text{lm}(ph \text{ rem } \mathcal{G}) = s\}|$$

We are going to define the order domain codes, and we can translate 2.2 in the language of semigroup.

**Definition 2.1.** *Let  $I \subseteq R$  be an ideal. Assume  $I$  possesses a Groebner basis  $\mathcal{G}$  s. t.*

(C1) *any  $g \in \mathcal{G}$  has exactly two monomials of highest weight in its support*

(C2) *no two monomials in  $\mathcal{N}_{<_W}(I)$  are of the same weight.*

*Then  $(I, <_W)$  satisfies the order domain conditions.*

If  $(I, <_W)$  satisfies the order domain conditions and  $L \subseteq R_q$ , the affine-variety code  $C(I, L)$  is called an **order domain code**.

**Example 2.1.** Let  $I = (x^3 + y^2 + y) \subseteq \mathbb{F}_4[x, y]$  and  $<_W$  given by  $w(x) = 2$ ,  $w(y) = 3$  and  $x <_{lex} y$ .  $\mathcal{G} = \{x^3 + y^2 + y\}$  is obviously a Groebner basis for  $I$ , and it is not difficult to verify that  $(I, <_W)$  satisfies the order domain condition. Then the code from the curve  $y^2 + y = x^3$  over  $\mathbb{F}_4$  is an order domain code. We observe that this code is called a Hermitian code since it is obtained by evaluating on the points of the Hermitian curve.

Observe that also other geometric codes, such as norm-trace codes, Reed-Solomon codes and Hyperbolic codes can be put into a form satisfying the order domain conditions ([Gei08]).

**Theorem 2.3.** Assume  $(I, <_W)$  satisfies the order domain conditions. The minimum distance of  $C(I, L)$  is at least

$$\min_{\alpha \in \mathcal{L}(L)} \sigma(w(\alpha))$$

and the minimum distance of  $C(I, L)^\perp$  is at least

$$\min\{\mu(w(h)) \mid h \in \mathcal{N}_{<}(I_q) \setminus \mathcal{L}(L)\} \geq \min\{\mu(\lambda) \mid \lambda \in \Gamma \setminus w(\mathcal{L}(L))\}$$

and so it is at least

$$\min\{\mu(\lambda) \mid \lambda \in \Gamma \setminus w(\mathcal{L}(L))\} \quad (1)$$

where  $\Gamma := w(\mathcal{N}_{<}(I))$  is the semigroup of the variable weights,  $\mu(\lambda) := |\{\alpha \in \Gamma \mid \exists \beta \in \Gamma \text{ s.t. } \alpha + \beta = \lambda\}|$ , for  $\lambda \in \Gamma$ , and  $\sigma(\alpha) := |\{\lambda \in w(\mathcal{N}(I_q)) \mid \exists \beta \in \Gamma \text{ s.t. } \alpha + \beta = \lambda\}|$ , for  $\alpha \in w(\mathcal{N}(I_q))$ .

One of the advantages of the Order Domain approach to geometric codes it is given by the bound on the distance (1) provided in previous theorem, since this is a bound that can be easily computed from the knowledge of the semigroup.

### 3 Computational improvements for Hilbert quasi-polynomials, with an application to Order Domain Codes

In this section we present an algorithm for an effective computation of Hilbert quasi-polynomials and we show how to exploit them for checking order domain conditions.

We have improved the Singular procedures showed in [CM16] to compute the Hilbert quasi-polynomial for rings  $\mathbb{K}[x_1, \dots, x_n]/I$ . Before showing the algorithm, we give some useful results.

**Proposition 3.1.** Let  $(R/I, <_W)$  be as usual and let  $\text{HP}_{R/I}^W(t) = \frac{h(t)}{g(t)}$ . Then it holds

$$\text{ri}^W(R/I) = \max\{0, \deg h(t) - \deg g(t) + 1\}$$

*Proof.* By Theorem 1.1 and Proposition 1.2 in [Sta07], it follows that since

$$\sum_{k \geq 0} H_{R/I}(k) t^k = \frac{h(t)}{g(t)}$$

with  $g(t) = \prod_{i=1}^n (1 - t^{w_i}) = \prod_{j=0}^{d-1} (1 - \zeta^j t)^{\alpha_j}$ , where  $\zeta$  is a primitive  $d$ th root of unity and  $\sum_{i=1}^n w_i = \sum_{j=0}^{d-1} \alpha_j = \deg g(t)$ , we obtain that, for all  $k \geq k_0$  with

$$k_0 = \begin{cases} 0 & \text{if } \deg h(t) < \deg g(t) \\ \deg h(t) - \deg g(t) + 1 & \text{if } \deg h(t) \geq \deg g(t) \end{cases},$$

the Hilbert function can be written as

$$H_{R/I}(k) = \sum_{i=0}^{d-1} S_i(k) \zeta^{ik}$$

where  $S_i(k)$  is a polynomial in  $k$  of degree less than  $\alpha_i$ . Then, by uniqueness of the Hilbert quasi-polynomial of period  $d$ , we deduce that the  $i$ th polynomial of  $P_{R/I}$  is given by  $P_i(t) := \sum_{j=0}^{d-1} S_j(t) \zeta^{ij}$  and that  $H_{R/I}(k) = P_i(k)$  for  $k \geq k_0$  and  $k \equiv i \pmod{d}$ .  $\square$

**Remark 3.1.** Since  $\text{HP}_R^W(t) = \frac{1}{\prod_{i=1}^n (1 - t^{w_i})}$ , the generalized regularity index of  $R$  is 0.

Thanks to the following two results we can recover  $H_{R/I}^W(k)$  from  $H_R^W(k)$  and  $P_{R/I}^{W'}$  from  $P_R^W$ , where  $W$  is obtained by  $W'$  dividing each  $w_i \in W'$  by  $\gcd(W')$ .

**Proposition 3.2.** Let  $\bar{I}$  be the initial ideal of  $I$  and  $\text{HP}_{R/\bar{I}}^W(t) = \frac{h(t)}{g(t)}$  the Hilbert-Poincaré series of  $R/\bar{I}$ , with  $h(t) = \sum_{i=0}^s h_i t^i$ . Then

$$H_{R/\bar{I}}^W(k) = \sum_{i=0}^s h_i H_R^W(k - i)$$

for all  $k \geq 0$ , with  $H_R^W(k) = 0$  for all  $k < 0$ .

*Proof.* Since

$$\text{HP}_R(t) = \sum_{k \geq 0} H_R(k) t^k = \frac{1}{\prod_{i=1}^n (1 - t^{w_i})}$$

we have

$$\text{HP}_{R/\bar{I}}(t) = \sum_{k \geq 0} H_{R/\bar{I}}(k) t^k = \frac{h(t)}{\prod_{i=1}^n (1 - t^{w_i})} = \left( \sum_{k \geq 0} H_R(k) t^k \right) \left( \sum_{j=0}^s h_j t^j \right) = \sum_{k \geq 0} \left( \sum_{j=0}^s h_j H_R(k - j) \right) t^k$$

where  $H_R(k) = 0$  for all  $k < 0$ . Therefore,  $H_{R/\bar{I}}(k) = \sum_{j=0}^s h_j H_R(k - j)$ , for all  $k \geq 0$ .  $\square$

**Lemma 3.1.** ([CM16]) Let  $W' := a \cdot W = [w'_1, \dots, w'_n]$  for some  $a \in \mathbb{N}_+$  and let  $\text{HP}_{R/I}^W(t) = \frac{\sum_{j=0}^s h_j t^j}{\prod_{i=1}^n (1 - t^{w_i})}$ . Then it holds:

1.  $P_{R/I}^W(k) = \sum_{j=0}^s h_j P_R^W(k - j)$  for all  $k \geq \text{ri}^W(R/I)$
2.  $P_R^{W'} = \{P'_0, \dots, P'_{ad-1}\}$  is such that

$$P'_i(x) = \begin{cases} 0 & \text{if } a \nmid i \\ P_{\frac{i}{a}}\left(\frac{x}{a}\right) & \text{if } a \mid i \end{cases}$$

### 3.1 Algorithm for computing Hilbert quasi-polynomials

Let  $(R/I, <_W)$  be as usual, we wish to compute its Hilbert quasi-polynomial  $P_{R/I}^W := \{P_0, \dots, P_{d-1}\}$ . Since we know some degree bounds for Hilbert quasi-polynomials, we can compute them by means of interpolation.

First off, let us consider  $I = (0)$  and  $W$  such that the  $w_i$ 's have no common factor. Each  $P_j$  has degree equal to  $n - 1$ , so, given  $j = 0, \dots, d - 1$ , we want to calculate  $P_j(x) := a_0 + a_1x + \dots + a_{n-1}x^{n-1}$  such that

$$P_j(k) = H_R(k) \quad \forall k \geq 0 \quad \text{and} \quad k \equiv j \pmod{d}$$

Therefore, let us consider the first  $n$  positive integers  $x_r$  such that  $P_j(x_r) = H_R(x_r)$

$$x_r := j + rd, \quad \text{for } r = 0, \dots, n - 1$$

By construction, the polynomial  $P_j(x)$  interpolates the points  $(x_r, H_R(x_r))$ .

Since we know the leading coefficient  $a_{n-1}$ , we can reduce the number of interpolation points  $x_r$ , and we get a system of linear equations in the coefficients  $a_i$ , with  $i = 0, \dots, n - 2$ . The system in matrix-vector form reads

$$\begin{bmatrix} 1 & x_0 & \dots & x_0^{n-2} \\ 1 & x_1 & \dots & x_1^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n-2} & \dots & x_{n-2}^{n-2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-2} \end{bmatrix} = \begin{bmatrix} H_R(x_0) - a_{n-1}x_0^{n-1} \\ H_R(x_1) - a_{n-1}x_1^{n-1} \\ \vdots \\ H_R(x_{n-2}) - a_{n-1}x_{n-2}^{n-1} \end{bmatrix}$$

We observe that for computing  $P_j$  the algorithm requires the computation of  $n - 1$  values of the Hilbert function, the construction of a Vandermonde matrix of dimension  $n - 1$  and its inversion. We have not yet shown how to calculate  $H_R(x_r)$ , for  $r = 0, \dots, n - 2$ .

Let  $k \in \mathbb{N}$ . The problem of calculating  $H_R(k)$  is equivalent to the problem of determining the number of partitions of an integer into elements of a finite set  $S := \{w_1, \dots, w_n\}$ , that is, the number of solutions in non-negative integers,  $\alpha_1, \dots, \alpha_n$ , of the equation

$$\alpha_1 w_1 + \dots + \alpha_n w_n = k$$

This problem was solved in [Syl82], [Gla09] and the solution is the coefficient of  $x^k$  in the following power series

$$\frac{1}{(1 - x^{w_1}) \dots (1 - x^{w_n})} \tag{2}$$

We are going to give an efficient method for getting the coefficient of  $x^k$  in the power series expansion of Equation (2). We refer to [Lee92] for a in-depth analysis on the power series expansion of a rational function. Let

$$g(x) = \prod_{i=1}^k (1 - \lambda_i x)^{\alpha_i} \quad \text{and} \quad f(x) = \prod_{i=k+1}^l (1 - \lambda_i x)^{\alpha_i}$$

be polynomials in  $\mathbb{C}[x]$ , where  $\alpha_1, \dots, \alpha_l$  are non-negative integers,  $\lambda_i$  are distinct and non-zero and the degree of  $f(x)$  is less than that of  $g(x)$ .

**Lemma 3.2.** *Let*

$$\frac{f(x)}{g(x)} = \sum_{n \geq 0} b(n)x^n$$

*the power series expansion of  $f(x)/g(x)$ . Then,*

$$b(n)n = \sum_{r=1}^n \left( \sum_{i=1}^k \alpha_i \lambda_i^r - \sum_{i=k+1}^l \alpha_i \lambda_i^r \right) b(n-r)$$

Let  $\zeta := \zeta_d$  be a primitive  $d$ th root of unity. Since

$$\prod_{i=1}^n (1 - x^{w_i}) = \prod_{j \in T_n} (1 - \zeta^j x)^n \prod_{j \in T_{n-1}} (1 - \zeta^j x)^{n-1} \cdots \prod_{j \in T_1} (1 - \zeta^j x)$$

for some pairwise disjoint subsets  $T_1, \dots, T_n \subseteq \{0, \dots, d-1\}$ , we can apply Lemma 3.2 with  $g(x) := \prod_{i=1}^k (1 - x^{w_i})$  and  $f(x) = 1$ . Observing that any  $j$  in  $\{0, \dots, d-1\}$  appears in exactly one of the  $T_i$ 's, say  $T_l$ , and that  $\zeta^j$  appears  $\iota$  times in the product, we obtain the following recursive formula for computing  $H_R(k)$

$$H_R(k) = \frac{1}{k} \sum_{r=1}^k \left[ \sum_{i=1}^n i \left( \sum_{j \in T_i} \zeta^{j^r} \right) \right] H_R(k-r) \quad (3)$$

It follows that if we know  $H_R(i)$  for all  $i = 1, \dots, k-1$ , we can easily compute  $H_R(k)$  by means of Equation (3).

Given an equation  $\alpha_1 w_1 + \dots + \alpha_n w_n = k$  the problem of counting the number of non-negative integer solutions  $\alpha_1, \dots, \alpha_n$  could be solved also using brute force. But, given  $k \in \mathbb{N}$ , to compute  $H_R^W(k)$  with brute force needs  $O(k^n)$  operations, whereas the procedure which we have implemented has a quadratic cost in  $k$ , in fact it needs  $O(nk^2)$  operations ([CM16]).

Up to now we have shown how to calculate  $P_R^W$ . For computing a Hilbert quasi-polynomial  $P_{R/I}^W = \{P_0, \dots, P_{d-1}\}$ , for any vector  $W$  and any homogeneous ideal  $I$  of  $R$ , the procedure computes first  $P_R^{W'}$ , where  $W'$  is obtained by dividing  $W$  by  $\gcd(w_1, \dots, w_n)$ , and then it produces  $P_{R/I}^W$  starting from  $P_R^W$ , using the relation between  $P_R^{W'}$  and  $P_{R/I}^W$  showed in Lemma 3.1.

### 3.2 Hilbert quasi-polynomials for order domain codes

To verify if a pair  $(I, <_W)$  satisfies the order domain condition (C1), it suffices to compute a Groebner basis of  $I$  w.r.t.  $<_W$  and, for each polynomial in the basis, to check the two monomials of highest weight. Whereas, for the condition (C2) we would need to study the ideal  $I$  or the semigroup  $\mathcal{N}_{<_W}(I)$ . We give an alternative and efficient way to establish if the condition (C2) is respected by  $(I, <_W)$ .

**Theorem 3.1.** *Let  $(I, <_W)$  be as usual. The following are equivalent:*

- (i)  $(I, <_W)$  satisfies the order domain condition (C2);
- (ii)  $H_{R/\bar{I}}(k) \in \{0, 1\}$  for all  $0 \leq k < \text{ri}^W(R/\bar{I})$  and each  $P_i$  in  $P_{R/\bar{I}}^W$  is the constant polynomial 0 or 1.



*Proof.* Since  $\mathcal{N}_{<_W}(I) = \{\text{lm}(f) \mid f \in R/\bar{I}\}$ , then  $H_{R/\bar{I}}(k) = |\{m \in \mathcal{N}_{<_W}(I) \mid w(m) = k\}|$ . We observe that condition (C2) can be equivalently formulated as requiring that for any possible weight there is at most one monomial in  $\mathcal{N}_{<_W}(I)$  enjoying that weight. In terms of Hilbert function, this conditions is then equivalent to

$$H_{R/\bar{I}}(k) \in \{0, 1\} \text{ for all } k \geq 0.$$

Recalling that  $H_{R/\bar{I}}$  will be eventually equal to the quasi-polynomial  $P_{R/\bar{I}}^W = \{P_0, \dots, P_{d-1}\}$ , our assertion follows.  $\square$

**Remark 3.2.** *The only requirement that each  $P_i$  in  $P_{R/\bar{I}}^W$  is the constant polynomial 0 or 1 does not imply (i), as we now know show. Let  $R = \mathbb{K}[x_1, x_2]$  with standard grading, and  $I = (x_1^2, x_1 x_2) \subseteq R$ . The Hilbert polynomial is the constant polynomial 1 but  $H_{R/I}(1) = 2$ , then  $I$  does not satisfy (C2). That is because  $H_{R/\bar{I}}(k) = P_{R/\bar{I}}^W(k)$  for all  $k \geq \text{ri}^W(R/\bar{I})$ , but  $P_{R/\bar{I}}^W$  does not give any information for  $k < \text{ri}^W(R/\bar{I})$ .*

**Corollary 3.1.** *Let  $(I, <_W)$  be as usual,  $\bar{I} = \text{in}(I)$  and  $\mathcal{G}$  a Groebner basis for  $I$ . If*

- *any  $g \in \mathcal{G}$  has exactly two monomials of highest weight in its support, and*
- *$H_{R/\bar{I}}(k) \in \{0, 1\}$  for all  $0 \leq k < \text{ri}^W(R/\bar{I})$  and each  $P_i$  in  $P_{R/\bar{I}}^W$  is the constant polynomial 0 or 1.*

*then  $(I, <_W)$  is an order domain.*

**Example 3.1.** *Return to Example 2.1. The Hilbert-Poincaré series of  $(R/\bar{I}, <_W)$  is given by  $\text{HP}_{R/\bar{I}}(t) = \frac{1-t^6}{(1-t^2)(1-t^3)}$ , then  $\text{ri}^W(R/\bar{I}) = 2$ . The Hilbert quasi-polynomial is  $P_{R/\bar{I}} = \{P_0, \dots, P_5\}$ , with  $P_i = 1$  for all  $i = 0, \dots, 5$ . Since  $H_{R/\bar{I}}(k) = P_{R/\bar{I}}(k)$  for all  $k \geq 2$ , we have only to check  $H_{R/\bar{I}}(1)$  which is obviously equal to 0, since 1 is a gap in the semigroup  $\Gamma = \langle 2, 3 \rangle$ .*

Now we are ready to describe the following

**Algorithm 3.1** (Check Order Domain).

- *Input:*  $(I, <_W)$ .
- *Output:*  $\text{IsOrderDomain} \in \{\text{True}, \text{False}\}$ .

1.  $\text{IsOrderDomain} := \text{False}$ ;
2. *Compute a Groebner basis  $\mathcal{G}$  for  $I$ ;*
3. *If  $w(\text{lt}(g)) = w(\text{lt}(g - \text{lt}(g)))$  for all  $g \in \mathcal{G}$ , then*
  - (a) *Let  $k_1 := \max\{\text{ri}^W(R/\bar{I}), d(n-1)\}$ ;*
  - (b) *Compute  $H_R(k)$  for all  $1 \leq k < k_1$ ;*
  - (c) *If  $H_{R/\bar{I}}(k) \in \{0, 1\}$  for all  $1 \leq k < k_1$  and each  $P_i \in P_{R/\bar{I}}$  is 0 or 1, then*  
 $\text{IsOrderDomain} := \text{True}$ .
4. *Return  $\text{IsOrderDomain}$ .*

**Remark 3.3** (Algorithm). *We recall the results necessary to the description of the algorithm.*

- *Line 3.* We check the condition (C1) of Corollary 3.1. If it is satisfied we test the second one.
- *Line 3a-3b.* We compute the first  $k_1$  values of the Hilbert function of the polynomial ring  $R$  using Equation (3), where  $k_1$  is the maximum between  $d(n-1)$ , that is the number of interpolation points, and the regularity index,  $\text{ri}^W(R/\bar{I})$ , which is known thanks to Proposition 3.1, where  $\deg h(t)$  is computed by Singular.
- *Line 3c.* Thanks to Proposition 3.2 we are able to check if  $H_{R/\bar{I}}(k) \in \{0, 1\}$  for the values  $0 \leq k < \text{ri}^W(R/\bar{I})$ . If the previous test does not fail we can compute, using algorithm showed in Section 3.1, the quasi-polynomial of  $R/\bar{I}$ , completing the test of the second condition described in Corollary 3.1.

## 4 Applications to codes from maximal curves

A maximal curve over a finite field  $\mathbb{F}_q$ , is a projective geometrically irreducible non-singular algebraic curve defined over  $\mathbb{F}_q$  whose number of  $\mathbb{F}_q$ -rational points attains the Hasse-Weil upper bound

$$q + 1 + 2g\sqrt{q};$$

where  $g$  is the genus of the curve. Maximal curves, especially those having large genus with respect to  $q$ , are known to be very useful in coding theory. In this section, we show some examples of affine-variety codes constructed over maximal curves, which are also order domains.

**Example 4.1.** Let  $\chi \subseteq \mathbb{P}^3$  is a  $\mathbb{F}_{49}$ -maximal curve of genus  $g = 7$  ([FGP12]) whose affine plane curve is

$$y^{16} = x(x+1)^6$$

Let  $I = (y^{16} - x(x+1)^6) \subseteq \mathbb{F}_{49}[x, y]$  and let  $<_W$  be given by  $w(x) = 16, w(y) = 7$  and  $x <_{\text{lex}} y$ . It is easy to verify that  $(I, <_W)$  satisfies the order domain condition (C1). Let us check condition (C2). Obviously,  $\bar{I} = (y^{16})$ . With our algorithm we have computed the Hilbert quasi-polynomial  $P_{R/\bar{I}} = \{P_0, \dots, P_{111}\}$ , and each  $P_i$  is equal to 1. With Singular we have computed  $\text{HP}_{R/\bar{I}}(t) = \frac{1-t^{112}}{(1-t^7)(1-t^{16})}$  and so  $\text{ri}^W(R/\bar{I}) = 90$ , which means that we are left with computing  $H_{R/\bar{I}}(k)$ , with  $0 < k < 90$ . With our recursive algorithm we can easily see that all obtained values are in  $\{0, 1\}$ . Then we can conclude that  $(I, <_W)$  is an order domain.

In the previous example we could avoid to compute the Hilbert function for values less than 90, thanks to the following lemma.

**Lemma 4.1.** Let  $I \subseteq \mathbb{K}[x_1, \dots, x_n]$  be an ideal such that its initial ideal  $\bar{I} \subseteq \mathbb{K}[x_1, \dots, x_{n-1}]$ , up to a reordering of variables. If each polynomial in the Hilbert quasi-polynomial of  $R/\bar{I}$  is 0 or 1, then  $H_{R/\bar{I}}(k) \in \{0, 1\}$  for all  $k \geq 0$ .

*Proof.* Suppose by contradiction  $H_{R/\bar{I}}(\tilde{k}) \notin \{0, 1\}$  for some  $\tilde{k} \geq 0$ , then there exist two distinct monomials  $m_1, m_2 \in R/\bar{I}$  with the same weight  $\tilde{k}$ . For all  $\alpha_n \geq 0$  also the monomials  $m_1 x_n^{\alpha_n}, m_2 x_n^{\alpha_n} \in R/\bar{I}$ , and they are distinct with the same weight  $\tilde{k} + \alpha_n w_n$ . In particular, it holds also for  $\tilde{k} + \alpha_n w_n \geq \text{ri}(R/\bar{I})$ , but this contradicts our hypothesis on the Hilbert quasi-polynomial of  $R/\bar{I}$ .  $\square$

**Remark 4.1.** If  $(I, <_W)$  satisfies the hypothesis of Lemma 4.1, then  $(I, <_W)$  is an order domain.

**Example 4.2.** Let  $q > 2$  be a prime power. The GK-curve, introduced by Giulietti and Korchmaros in [GK09], is the curve  $\mathcal{C}_3 \subseteq \mathbb{P}^3$  defined over  $\mathbb{F}_{q^6}$  by the affine equations

$$v^{q+1} = u^q + u \quad \text{and} \quad w^{\frac{q^3+1}{q+1}} = v h(u)$$

where  $h(u) = (u^q + u)^{q-1} - 1$ . This curve is maximal over  $\mathbb{F}_{q^6}$  and it is so far the only known example of a maximal curve which cannot be dominated by the Hermitian curve. It turns out in [GGS10] that  $\mathcal{C}_3$  can also be defined by the equations

$$v^{q+1} = u^q + u \quad \text{and} \quad w^{\frac{q^3+1}{q+1}} = v^{q^2} - v$$

We are going to investigate  $\mathcal{C}_3$ , with an opportune generalized weighted degree ordering  $<_W$ , defines an order domain. Let  $q = 3$ , and  $I = (v^4 - u^3 - u, w^7 - v^9 + v) \in \mathbb{F}_{3^6}[u, v, w]$  with  $u <_{lex} v <_{lex} w$ . Obviously,  $\mathcal{G} = \{v^4 - u^3 - u, w^7 - v^9 + v\}$  is a Groebner basis for  $I$  and  $\tilde{I} = (v^4, w^7)$ . In order to satisfy condition (C1), we set  $W = [28, 21, 27]$ . Note that, up to a constant factor,  $W$  is unique. Since  $\text{lcm}(28, 21, 27) = 756$ , we have computed the Hilbert quasi-polynomial  $P_{R/\tilde{I}} = \{P_0, \dots, P_{755}\}$  of  $(R/\tilde{I}, <_W)$ . Since each  $P_i$  turns out to be 1, thanks to the remark above, we can conclude that  $(I, <_W)$  is an order domain.

## 5 Conclusions and further comments

Our algorithm 3.1 allows to decide effectively if an affine-variety code can be seen as an order-domain code w.r.t. a generalized weighted degree ordering. Our algorithm can indeed solve some interesting examples, as we have shown, and as such we believe it can become a convenient tools for coding theorists working with geometric codes.

We would like now to point out an interesting aspect of our approach. For our computation of Hilbert quasi-polynomials it is essential to work in a characteristic-0 field. In our applications we have positive characteristic fields for the Order Domain Codes. However, the actual field  $\mathbb{F}_q$  matters only for the computation of the Groebner basis of  $I$ . Once the Groebner basis has been obtained, the monomials under the Hilbert staircase are the same for any field and so we can consider the leading monomials of the obtained Groebner basis as if they were on a characteristic-zero field.

We see at least three paths to follow in order to increase the impact of our approach:

- An advantage of Order Domain Codes (with respect to more traditional codes over curves) is that we can build them on higher dimensions variety, e.g. the surface in Example 51 [AG08]. The higher dimension requires more advanced weight orderings that cannot be tackled by our present theory and algorithms. Therefore, we believe that this extension is natural and worth studying.
- Our algorithm is well-suited when a given variety has been chosen to build the code. Often it happens that we know an infinite family of varieties that would be ideal to build codes on them, e.g. known families of maximal curves. In

this situation we would need to adapt our computational approach to a more theoretical one, in order to use the core ideas of our algorithms for generating general proofs.

- We would like to use Hilbert quasi-polynomials to deduce some code-theoretical properties of the corresponding Order Domain Codes, but it is not immediately clear to us how this could be achieved.

We will follow these paths and we are open to collaborations.

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